Chained Clauser-Horne-Shimony-Holt inequality for Hardy's ladder test of nonlocality

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Abstract Relativistic causality forbids superluminal signaling between distant observers. By exploiting the non-signaling principle, we derive the exact relationship between the chained Clauser-Horne-Shimony-Holt sum of correlations CHSH_K and the success probability P_K associated with Hardy's ladder test of nonlocality for two qubits and $K + 1$ observables per qubit. Then, by invoking the Tsirelson bound for CHSH_K, the derived relationship allows us to establish an upper limit on P_K. In addition, we draw the connection between CHSH_K and the chained version of the Clauser-Horne (CH) inequality.

1 Introduction

In 1992, Lucien Hardy [1] gave a new proof of nonlocality without inequalities for two particles that only requires a total of four dimensions in Hilbert space. He further showed [2] that this proof works for all pure entangled states of two two-state systems or qubits except for the maximally entangled state. Hardy's proof [2] (which concerns two observables for each qubit) was subsequently extended to the case in which there are $K + 1$ available dichotomic observables per qubit— $A_0, A_1, \ldots, A_K$ for qubit A and $B_0, B_1, \ldots, B_K$ for qubit B, where $K = 1, 2, 3, \ldots$ [3,4]. We will refer to this latter proof as Hardy's ladder test of nonlocality. In order to have a contradiction between quantum mechanics and local realism in Hardy's ladder scenario, the observables A_k and $B_{k'}$ $(k, k' = 0, 1, \ldots, K)$ are required to

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satisfy the following conditions [3,4]

$$P_K = P(A_K = +1, B_K = +1) \neq 0, \tag{1}$$

$$P(A_k = +1, B_{k-1} = -1) = 0, \quad \text{for } k = 1 \text{ to } K, \tag{2}$$

$$P(A_{k-1} = -1, B_k = +1) = 0, \quad \text{for } k = 1 \text{ to } K, \tag{3}$$

$$P(A_0 = +1, B_0 = +1) = 0, \tag{4}$$

where $P(A_k = i, B_{k'} = j)$ is the joint conditional probability of obtaining the result i when measuring A_k on qubit A and obtaining the result j when measuring $B_{k'}$ on qubit B ($i, j = \pm 1$). According to a local realistic (LR) theory, fulfillment of the $2K + 1$ conditions in (2)-(4) necessarily implies that $P_K = 0$. Quantum-mechanically, however, we can have $P_K \neq 0$ while all the other conditions in (2)-(4) are satisfied. The success probability P_K of Hardy's nonlocality argument is sometimes known as the "Hardy fraction."

It should be noted that the conditions (2)-(4) could not be satisfied strictly in practical experiments due to the difficulty of experimentally measuring a null event. Indeed, *even* with perfect measurement apparatus it is not possible to achieve a true "zero" value for the various probabilities because the number of measurements in a real experiment is necessarily finite [5,6]. Moreover, as pointed out in Ref. [4], to test experimentally Hardy's conditions (1)-(4), inequalities are necessary in order to make sure that the errors do not wash out the logical conundrum faced by local realism. One such suitable inequality for Hardy's ladder test is the chained Clauser-Horne-Shimony-Holt-type (CHSH) inequality [7,8,9]

$$\left| \sum_{k=1}^{K} E(A_k, B_{k-1}) + \sum_{k=1}^{K} E(A_{k-1}, B_k) + E(A_K, B_K) - E(A_0, B_0) \right| \overset{\text{LR}}{\leq} 2K, \tag{5}$$

which holds for any LR theory, with the correlation function $E(A_k, B_{k'})$ defined by $E(A_k, B_{k'}) = P^+(A_k, B_{k'}) - P^-(A_k, B_{k'})$, where

$$P^+(A_k, B_{k'}) = P(A_k = +1, B_{k'} = +1) + P(A_k = -1, B_{k'} = -1),$$

$$P^-(A_k, B_{k'}) = P(A_k = +1, B_{k'} = -1) + P(A_k = -1, B_{k'} = +1),$$

and $P^+(A_k, B_{k'}) + P^-(A_k, B_{k'}) = 1$. Notice that the $2K + 2$ pairs of observables (A_0, B_0), (A_K, B_K), (A_{k-1}, B_k), (A_k, B_{k-1}), $k = 1, 2, \ldots, K$, occurring on the left-hand side of inequality (5) are precisely those in Eqs. (1)-(4). Evidently, for $K = 1$, the inequality (5) reduces to the original CHSH inequality [10]. It is well known that the maximum quantum violation of the CHSH inequality is given by Tsirelson's bound $2\sqrt{2}$ [11]. Furthermore, Wehner [9] showed that the corresponding Tsirelson bound for the chained CHSH inequality (5) is given by

$$\left| \sum_{k=1}^{K} E(A_k, B_{k-1}) + \sum_{k=1}^{K} E(A_{k-1}, B_k) + E(A_K, B_K) - E(A_0, B_0) \right|$$

$$\overset{\text{QM}}{\leq} 2(K+1) \cos \frac{\pi}{2(K+1)}. \tag{6}$$

Let us denote the sum of correlations on the left-hand side of either (5) or (6) as CHSH_K, that is,

$$\mathrm{CHSH}_K \equiv \sum_{k=1}^{K} E(A_k, B_{k-1}) + \sum_{k=1}^{K} E(A_{k-1}, B_k) + E(A_K, B_K) - E(A_0, B_0). \quad (7)$$

In this paper (Sec. 2), we show that, for the case in which the conditions in Eqs. (2)-(4) are met, the whole CHSH_K expression (7) can be written in terms of the Hardy fraction P_K through the simple relation

$$\mathrm{CHSH}_K = 2K + 4P_K, \quad K = 1, 2, 3, \ldots . \quad (8)$$

Remarkably, as we will see, relation (8) follows as a consequence of the non-signaling (NS) principle alone. Every probabilistic theory respecting the NS principle (including quantum mechanics) should therefore comply with relation (8).[1] This relation was already obtained elsewhere for the simplest case $K = 1$ [12]. It has also been derived independently (for $K = 1$) by Xiang in Ref. [13]. Relation (8) embodies the precise connection between, on the one hand, Hardy's ladder test of nonlocality based on Eqs. (1)-(4) and, on the other hand, the test of nonlocality based on the generalized CHSH inequality (5).

An immediate implication of relation (8) is that, when the Hardy conditions (1)-(4) are fulfilled, the inequality (5) is violated by an amount $2K + 4P_K \leq 2K$, or $4P_K \leq 0$. It is important to note that this amount is four times bigger than that obtained for the chained version of the Clauser-Horne (CH) inequality [4, 5, 6, 14, 15]

$$P(A_K = +1, B_K = +1) - P(A_0 = +1, B_0 = +1)$$
$$- \sum_{k=1}^{K} \left[P(A_k = +1, B_{k-1} = -1) + P(A_{k-1} = -1, B_k = +1) \right] \overset{\mathrm{LR}}{\leq} 0,$$

which is the inequality commonly used in the experimental realizations of Hardy's ladder test of nonlocality (see, for example, Refs. [4, 16, 17, 18]).

In Sec. 3, we will establish (see Eq. (24) below) the relationship between CHSH_K and the sum of probabilities on the left-hand side of the above CH-type inequality for the general case in which the probabilities are constrained only by the NS principle. The relation (8) and its generalization (24) are the main results of this paper.

Moreover, combining the relation (8) and the Tsirelson bound (6) gives us the following upper limit for P_K

$$P_K \overset{\mathrm{QM}}{\leq} \frac{1}{4} \left[2(K+1) \cos \frac{\pi}{2(K+1)} - 2K \right] \equiv L_K. \quad (9)$$

Therefore, if quantum mechanics is correct, the Hardy fraction has to be bounded above by the upper limit L_K in Eq. (9). In particular, for the first five values of

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[1] In Sec. 3, it will be verified that, in fact, the quantum predictions satisfy relation (8).
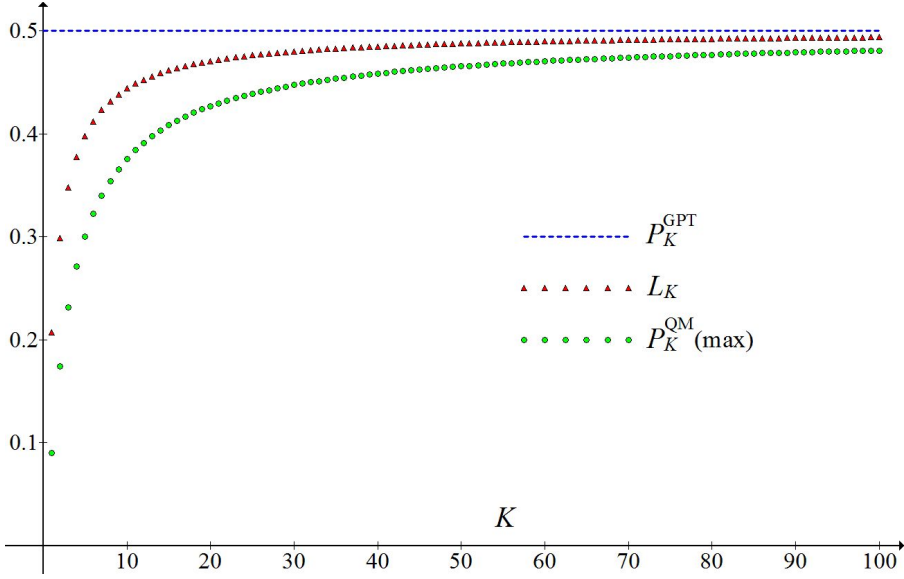
Fig. 1 Plot of the upper limit L_K (red triangles) and the quantum prediction $P_K^{\mathrm{QM}}(\mathrm{max})$ (green circles) against K (for $K = 1$ to 100) of the success probability of Hardy's nonlocality argument. The horizontal line $P_K^{\mathrm{GPT}} = 0.5$ represents the maximum success probability allowed by a generalized probabilistic theory.

K, from (9) we obtain

$$P_1 \le \tfrac{1}{2}(\sqrt{2} - 1) \approx 0.207106, \qquad\qquad P_2 \le \tfrac{1}{4}(3\sqrt{3} - 4) \approx 0.299038,$$

$$P_3 \le \sqrt{2 + \sqrt{2}} - \tfrac{3}{2} \approx 0.347759, \qquad P_4 \le \tfrac{5}{8}\sqrt{10 + 2\sqrt{5}} - 2 \approx 0.377641,$$

$$P_5 \le \tfrac{1}{4}\big(3(\sqrt{2} + \sqrt{6}) - 10\big) \approx 0.397777.$$

In Fig. 1, we have plotted the upper limit L_K for $K = 1$ to 100. For comparison, we have also plotted the maximum success probability $P_K^{\mathrm{QM}}(\mathrm{max})$ achieved by quantum mechanics. For a given K, this latter probability is obtained by maximizing the function $P_K^{\mathrm{QM}}(x)$ with respect to x in the interval $0 \le x \le 1$, where [3, 4, 19]

$$P_K^{\mathrm{QM}}(x) = \frac{x^2}{1 + x^2}\left(\frac{1 - x^{2K}}{1 + x^{2K+1}}\right)^2.$$

The asymptotic value of both L_K and $P_K^{\mathrm{QM}}(\mathrm{max})$ is $L_K = P_K^{\mathrm{QM}}(\mathrm{max}) = 0.5$ in the limit $K \to \infty$. Quantum-mechanically, the absolute maximum $P_K^{\mathrm{QM}}(\mathrm{max}) = 0.5$ is realized for $K \to \infty$ and a state that is close to maximally entangled ($x \to 1$) [3, 4]. From relation (8) it follows that, as $P_K^{\mathrm{QM}}(\mathrm{max}) \to 0.5$, the CHSH$_K$ expression (7) approaches the algebraic limit $2K + 2$.[2] Furthermore, it is known [20] that a generalized probabilistic theory (GPT) adhering to the NS principle allows for

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[2] Incidentally, it is worth pointing out that, in this limit, a direct ("all or nothing") contradiction between quantum mechanics and local realism emerges in Hardy's ladder scenario [19].

a maximum Hardy's fraction equal to 0.5 *independently* of the value of K. This is indicated in Fig. 1 by the horizontal line $P_K^{\mathrm{GPT}} = 0.5$.[3] Indeed, the following extremal NS probability distribution [20,21]

$$P(A_k = i, B_{k'} = j) = \begin{cases} \frac{1}{2}, \text{ for } i = j \text{ and } \forall k, k' \in \{0, 1, \ldots, K\} \text{ except for } k = k' = 0; \\ 0, \text{ for } i = j \text{ and } k = k' = 0; \\ 0, \text{ for } i \neq j \text{ and } \forall k, k' \in \{0, 1, \ldots, K\} \text{ except for } k = k' = 0; \\ \frac{1}{2}, \text{ for } i \neq j \text{ and } k = k' = 0, \end{cases}$$

satisfies all the Hardy conditions (1)-(4) with $P_K = 0.5$, as well as the requirements of normalization and non-negativity, and gives the maximum algebraic bound (namely, $2K + 2$) of inequality (5). For $K = 1$, it corresponds to the Popescu-Rohrlich-type correlations [22] leading to the maximum algebraic violation (namely, 4) of the CHSH inequality while preserving relativistic causality (see Ref. [23] for several variants of the above extremal distribution for $K = 1$).

2 Chained CHSH Inequality for Hardy's Ladder Test of Nonlocality

We devote this section to prove relation (8). This is done by employing certain judiciously chosen relationships imposed by the NS principle. For the Hardy ladder scenario, this principle requires that the marginal probability $P(A_k = i)$ $[P(B_{k'} = j)]$ of obtaining the result i $[j]$ in a measurement of A_k $[B_{k'}]$ on qubit A $[B]$ is independent of which measurement $B_0, B_1, \ldots, B_K$ $[A_0, A_1, \ldots, A_K]$ is performed on the distant qubit B $[A]$. In terms of joint probabilities, this requirement amounts to the following set of conditions:

$$\sum_{j=\pm 1} P(A_k = i, B_0 = j) = \sum_{j=\pm 1} P(A_k = i, B_1 = j)$$
$$= \ldots = \sum_{j=\pm 1} P(A_k = i, B_K = j) \quad \forall k, i, \qquad (10)$$

$$\sum_{i=\pm 1} P(A_0 = i, B_{k'} = j) = \sum_{i=\pm 1} P(A_1 = i, B_{k'} = j)$$
$$= \ldots = \sum_{i=\pm 1} P(A_K = i, B_{k'} = j) \quad \forall k', j, \qquad (11)$$

where $k, k' = 0, 1, \ldots, K$ and $i, j = \pm 1$.

To prove relation (8), we first rewrite the CHSH_K expression (7) in the equivalent form

$$\mathrm{CHSH}_K = 2K + 2P^+(A_K, B_K) - 2P^+(A_0, B_0)$$
$$- 2\sum_{k=1}^{K} P^-(A_k, B_{k-1}) - 2\sum_{k=1}^{K} P^-(A_{k-1}, B_k).$$

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[3]   Note that, from relation (8), the limit $P_K = 0.5$ can never be surpassed since this would imply that $\mathrm{CHSH}_K > 2K + 2$, which is impossible by the very definition of CHSH_K.

For the case in which the conditions in (2)-(4) are fulfilled, the above expression reduces to

$$\text{CHSH}_K = 2K + 2P_K + 2P(A_K = -1, B_K = -1) - 2P(A_0 = -1, B_0 = -1)$$
$$- 2\sum_{k=1}^{K} P(A_k = -1, B_{k-1} = +1) - 2\sum_{k=1}^{K} P(A_{k-1} = +1, B_k = -1).$$

Therefore, in order to prove relation (8), it suffices to show that

$$P(A_K = -1, B_K = -1) = P_K + P(A_0 = -1, B_0 = -1)$$
$$+ \sum_{k=1}^{K} P(A_k = -1, B_{k-1} = +1) + \sum_{k=1}^{K} P(A_{k-1} = +1, B_k = -1). \quad (12)$$

In what follows we show that relation (12) is indeed fulfilled for $K = 1$ and 2, and then we establish the result generally. In the rest of this section, we employ the abbreviated notation $P_{kk'}^{ij}$ to refer to the joint probability $P(A_k = i, B_{k'} = j)$.

2.1 Case $K = 1$

For $K = 1$, the NS conditions in Eqs. (10) and (11) read as follows:

$$\begin{aligned}
P_{00}^{++} + P_{00}^{+-} &= P_{01}^{++} + P_{01}^{+-} \\
P_{10}^{++} + P_{10}^{+-} &= P_{11}^{++} + P_{11}^{+-} \\
P_{00}^{-+} + P_{00}^{--} &= P_{01}^{-+} + P_{01}^{--} \\
P_{10}^{-+} + P_{10}^{--} &= P_{11}^{-+} + P_{11}^{--} \\
P_{00}^{++} + P_{00}^{-+} &= P_{10}^{++} + P_{10}^{-+} \\
P_{01}^{++} + P_{01}^{-+} &= P_{11}^{++} + P_{11}^{-+} \\
P_{00}^{+-} + P_{00}^{--} &= P_{10}^{+-} + P_{10}^{--} \\
P_{01}^{+-} + P_{01}^{--} &= P_{11}^{+-} + P_{11}^{--}.
\end{aligned} \quad (13)$$

Furthermore, Hardy's conditions (2)-(4) for $K = 1$ mean that

$$P_{00}^{++} = P_{01}^{-+} = P_{10}^{+-} = 0. \quad (14)$$

Hence using (14) in (13), we readily obtain

$$\begin{aligned}
P_{00}^{+-} &= P_{01}^{++} + P_{01}^{+-} \\
P_{10}^{++} &= P_{11}^{++} + P_{11}^{+-} \\
P_{01}^{--} &= P_{00}^{-+} + P_{00}^{--} \\
P_{11}^{--} + P_{11}^{-+} &= P_{10}^{-+} + P_{10}^{--} \\
P_{00}^{-+} &= P_{10}^{++} + P_{10}^{-+} \\
P_{01}^{++} &= P_{11}^{++} + P_{11}^{-+} \\
P_{10}^{--} &= P_{00}^{+-} + P_{00}^{--} \\
P_{11}^{--} + P_{11}^{+-} &= P_{01}^{+-} + P_{01}^{--}.
\end{aligned} \quad (15)$$

Summing all eight relationships in (15) and simplifying gives $P_{11}^{--} = P_{11}^{++} + P_{00}^{--} + P_{10}^{-+} + P_{01}^{+-}$, which is just relation (12) for $K = 1$.

2.2 Case $K = 2$

For $K = 2$, the NS conditions in Eqs. (10) and (11) imply that

$$
\begin{aligned}
P_{00}^{++} + P_{00}^{+-} &= P_{01}^{++} + P_{01}^{+-} = P_{02}^{++} + P_{02}^{+-} \\
P_{10}^{++} + P_{10}^{+-} &= P_{11}^{++} + P_{11}^{+-} = P_{12}^{++} + P_{12}^{+-} \\
P_{20}^{++} + P_{20}^{+-} &= P_{21}^{++} + P_{21}^{+-} = P_{22}^{++} + P_{22}^{+-} \\
P_{00}^{-+} + P_{00}^{--} &= P_{01}^{-+} + P_{01}^{--} = P_{02}^{-+} + P_{02}^{--} \\
P_{10}^{-+} + P_{10}^{--} &= P_{11}^{-+} + P_{11}^{--} = P_{12}^{-+} + P_{12}^{--} \\
P_{20}^{-+} + P_{20}^{--} &= P_{21}^{-+} + P_{21}^{--} = P_{22}^{-+} + P_{22}^{--} \\
P_{00}^{++} + P_{00}^{-+} &= P_{10}^{++} + P_{10}^{-+} = P_{20}^{++} + P_{20}^{-+} \\
P_{01}^{++} + P_{01}^{-+} &= P_{11}^{++} + P_{11}^{-+} = P_{21}^{++} + P_{21}^{-+} \\
P_{02}^{++} + P_{02}^{-+} &= P_{12}^{++} + P_{12}^{-+} = P_{22}^{++} + P_{22}^{-+} \\
P_{00}^{+-} + P_{00}^{--} &= P_{10}^{+-} + P_{10}^{--} = P_{20}^{+-} + P_{20}^{--} \\
P_{01}^{+-} + P_{01}^{--} &= P_{11}^{+-} + P_{11}^{--} = P_{21}^{+-} + P_{21}^{--} \\
P_{02}^{+-} + P_{02}^{--} &= P_{12}^{+-} + P_{12}^{--} = P_{22}^{+-} + P_{22}^{--},
\end{aligned}
\tag{16}
$$

while Hardy's conditions (2)-(4) for $K = 2$ are

$$
P_{00}^{++} = P_{01}^{-+} = P_{10}^{+-} = P_{12}^{-+} = P_{21}^{+-} = 0.
\tag{17}
$$

Then, taking into account (17), we pick out the following subset of relationships among those in the set (16):

$$
\begin{aligned}
P_{00}^{+-} &= P_{01}^{++} + P_{01}^{+-} \\
P_{10}^{++} &= P_{12}^{++} + P_{12}^{+-} \\
P_{21}^{++} &= P_{22}^{++} + P_{22}^{+-} \\
P_{01}^{--} &= P_{00}^{-+} + P_{00}^{--} \\
P_{12}^{--} &= P_{10}^{-+} + P_{10}^{--} \\
P_{22}^{--} + P_{22}^{-+} &= P_{21}^{-+} + P_{21}^{--} \\
P_{00}^{-+} &= P_{10}^{++} + P_{10}^{-+} \\
P_{01}^{++} &= P_{21}^{++} + P_{21}^{-+} \\
P_{12}^{++} &= P_{22}^{++} + P_{22}^{-+} \\
P_{10}^{--} &= P_{00}^{+-} + P_{00}^{--} \\
P_{21}^{--} &= P_{01}^{+-} + P_{01}^{--} \\
P_{22}^{--} + P_{22}^{+-} &= P_{12}^{+-} + P_{12}^{--}.
\end{aligned}
\tag{18}
$$

Summing all twelve relationships in (18) and simplifying, we get $P_{22}^{--} = P_{22}^{++} + P_{00}^{--} + P_{10}^{-+} + P_{21}^{-+} + P_{01}^{+-} + P_{12}^{+-}$, which is just relation (12) for $K = 2$.

2.3 The General Case

The above proofs for $K = 1$ and 2 generalize straightforwardly to an arbitrary number $K + 1$ of observables per qubit. To show this, we write the NS conditions (10) and (11) in the expanded form

$$
K+1 \begin{cases}
P_{00}^{++} + P_{00}^{+-} = P_{01}^{++} + P_{01}^{+-} = P_{02}^{++} + P_{02}^{+-} = \ldots = P_{0K}^{++} + P_{0K}^{+-} \\
P_{10}^{++} + P_{10}^{+-} = P_{11}^{++} + P_{11}^{+-} = P_{12}^{++} + P_{12}^{+-} = \ldots = P_{1K}^{++} + P_{1K}^{+-} \\
\vdots \\
P_{K0}^{++} + P_{K0}^{+-} = P_{K1}^{++} + P_{K1}^{+-} = P_{K2}^{++} + P_{K2}^{+-} = \ldots = P_{KK}^{++} + P_{KK}^{+-}
\end{cases}
$$

$$
K+1 \begin{cases}
P_{00}^{-+} + P_{00}^{--} = P_{01}^{-+} + P_{01}^{--} = P_{02}^{-+} + P_{02}^{--} = \ldots = P_{0K}^{-+} + P_{0K}^{--} \\
P_{10}^{-+} + P_{10}^{--} = P_{11}^{-+} + P_{11}^{--} = P_{12}^{-+} + P_{12}^{--} = \ldots = P_{1K}^{-+} + P_{1K}^{--} \\
\vdots \\
P_{K0}^{-+} + P_{K0}^{--} = P_{K1}^{-+} + P_{K1}^{--} = P_{K2}^{-+} + P_{K2}^{--} = \ldots = P_{KK}^{-+} + P_{KK}^{--}
\end{cases}
$$

$$
K+1 \begin{cases}
P_{00}^{++} + P_{00}^{-+} = P_{10}^{++} + P_{10}^{-+} = P_{20}^{++} + P_{20}^{-+} = \ldots = P_{K0}^{++} + P_{K0}^{-+} \\
P_{01}^{++} + P_{01}^{-+} = P_{11}^{++} + P_{11}^{-+} = P_{21}^{++} + P_{21}^{-+} = \ldots = P_{K1}^{++} + P_{K1}^{-+} \\
\vdots \\
P_{0K}^{++} + P_{0K}^{-+} = P_{1K}^{++} + P_{1K}^{-+} = P_{2K}^{++} + P_{2K}^{-+} = \ldots = P_{KK}^{++} + P_{KK}^{-+}
\end{cases} \tag{19}
$$

$$
K+1 \begin{cases}
P_{00}^{+-} + P_{00}^{--} = P_{10}^{+-} + P_{10}^{--} = P_{20}^{+-} + P_{20}^{--} = \ldots = P_{K0}^{+-} + P_{K0}^{--} \\
P_{01}^{+-} + P_{01}^{--} = P_{11}^{+-} + P_{11}^{--} = P_{21}^{+-} + P_{21}^{--} = \ldots = P_{K1}^{+-} + P_{K1}^{--} \\
\vdots \\
P_{0K}^{+-} + P_{0K}^{--} = P_{1K}^{+-} + P_{1K}^{--} = P_{2K}^{+-} + P_{2K}^{--} = \ldots = P_{KK}^{+-} + P_{KK}^{--}
\end{cases}
$$

with a total of $4(K+1)$ rows and K equals signs in each row. Furthermore, Hardy's conditions in (2)-(4) read as

$$
P_{00}^{++} = P_{01}^{-+} = P_{10}^{+-} = P_{12}^{-+} = P_{21}^{+-} = \ldots = P_{K-1,K}^{-+} = P_{K,K-1}^{+-} = 0. \tag{20}
$$

Then, using the $2K + 1$ conditions (20) in (19), we can extract the following appropriate set of $4(K + 1)$ relationships:

$$
K+1 \begin{cases}
P_{00}^{+-} = P_{01}^{++} + P_{01}^{+-} \\
P_{10}^{++} = P_{12}^{++} + P_{12}^{+-} \\
\vdots \\
P_{KK-1}^{++} = P_{KK}^{++} + P_{KK}^{+-}
\end{cases}
$$

$$
K+1 \begin{cases}
P_{01}^{--} = P_{00}^{-+} + P_{00}^{--} \\
P_{12}^{--} = P_{10}^{-+} + P_{10}^{--} \\
\vdots \\
P_{K-1K}^{--} = P_{K-1K-2}^{-+} + P_{K-1K-2}^{--} \\
P_{KK}^{--} + P_{KK}^{-+} = P_{KK-1}^{-+} + P_{KK-1}^{--}
\end{cases} \tag{21a}
$$

and

$$K+1 \begin{cases} P_{00}^{-+} = P_{10}^{++} + P_{10}^{-+} \\ P_{01}^{++} = P_{21}^{++} + P_{21}^{-+} \\ \vdots \\ P_{K-1K}^{++} = P_{KK}^{++} + P_{KK}^{-+} \end{cases}$$

$$K+1 \begin{cases} P_{10}^{--} = P_{00}^{+-} + P_{00}^{--} \\ P_{21}^{--} = P_{01}^{+-} + P_{01}^{--} \\ \vdots \\ P_{KK-1}^{--} = P_{K-2K-1}^{+-} + P_{K-2K-1}^{--} \\ P_{KK}^{--} + P_{KK}^{+-} = P_{K-1K}^{+-} + P_{K-1K}^{--} \end{cases} \tag{21b}$$

such that the right-hand side (rhs) of each of the $2K + 2$ relationships in both subsets (21a) and (21b) contains exactly one of the $2K + 2$ joint probabilities appearing on the rhs of relation (12), with each subset exhausting all these $2K + 2$ probabilities. In particular, the probability P_{KK}^{++} $[P_{00}^{--}]$ occurs on the rhs of the $(K+1)$-th $[(K+2)$-th] relationship in each subset. On the other hand, the probability P_{KK}^{--} on the left-hand side (lhs) of relation (12) occurs on the lhs of the $(2K+2)$-th relationship of both subsets (21a) and (21b). Moreover, such subsets exhibit the following additional features: (i) Every single probability $P_{kk'}^{ij}$ on the lhs of the $2K + 2$ relationships in (21a) [(21b)] (leaving aside the distinguished probability P_{KK}^{--}) has the counterpart $P_{k'k}^{ji}$ in the rhs of another relationship of the same subset (21a) [(21b)]; (ii) The corresponding r-th relationships in (21a) and (21b), $r = 1, 2, \ldots, 2K + 2$, can be transformed into each other by swapping the superscripts $i \leftrightarrow j$ and the subscripts $k \leftrightarrow k'$ of all the probabilities $P_{kk'}^{ij}$ entering the given r-th relationships. With all these ingredients at hand, it is not difficult to see that, on summing all $4(K+1)$ relationships in (21a) and (21b), and simplifying, we end up with relation (12).

3 Concluding Remarks

As pointed out in Sec. 1, the quantum mechanical predictions should satisfy relation (8), since it is a consequence of the NS principle. Next we confirm that, indeed, the joint probabilities predicted by quantum mechanics for Hardy's ladder scenario satisfy the equivalent relation (12).

Consider two qubits A and B in the generic pure entangled state

$$|\Psi\rangle = \frac{x}{\sqrt{1+x^2}}|+\rangle_A|+\rangle_B - \frac{1}{\sqrt{1+x^2}}|-\rangle_A|-\rangle_B, \tag{22}$$

where $\{|+\rangle_A, |-\rangle_A\}$ $(\{|+\rangle_B, |-\rangle_B\})$ is an arbitrary orthonormal basis in the state space of qubit A (B), and $0 \leq x \leq 1$. Note that $x = 0$ ($x = 1$) corresponds to the product (maximally entangled) state. For the state (22) and for an optimal choice of observables, the quantum prediction (subject to the fulfillment of conditions

(2)-(4)) for the various joint probabilities in relation (12) is [3, 4, 19]

$$P^{\mathrm{QM}}(A_0 = -1, B_0 = -1) = \frac{(1-x)^2}{1+x^2},$$

$$P^{\mathrm{QM}}(A_K = -1, B_K = -1) = \frac{1}{1+x^2}\left(\frac{1-x^{2K+2}}{1+x^{2K+1}}\right)^2,$$

$$P^{\mathrm{QM}}(A_K = +1, B_K = +1) = \frac{x^2}{1+x^2}\left(\frac{1-x^{2K}}{1+x^{2K+1}}\right)^2,$$

and, for $k = 1, 2, \ldots, K$,

$$P^{\mathrm{QM}}(A_k = -1, B_{k-1} = +1) = P^{\mathrm{QM}}(A_{k-1} = +1, B_k = -1)$$

$$= \frac{(1-x^2)^2}{x(1+x^2)}\frac{x^{2k}}{(1+x^{2k-1})(1+x^{2k+1})}.$$

Substituting these expressions into relation (12) and simplifying it, eventually yields the identity

$$\sum_{k=1}^{K}\frac{x^{2k}}{(1+x^{2k-1})(1+x^{2k+1})} = \frac{x^2(x^{2K}-1)}{(1+x)(x^2-1)(1+x^{2K+1})},$$

which can be easily proved by mathematical induction on k. It is worth noting that, by using the auxiliary identity $x^{2K} - 1 = (x^2 - 1)\sum_{j=0}^{K-1} x^{2j}$, the above identity can be rewritten as

$$\sum_{k=1}^{K}\frac{x^{2k}}{(1+x^{2k-1})(1+x^{2k+1})} = \frac{1}{(1+x)(1+x^{2K+1})}\sum_{k=1}^{K}x^{2k},$$

which holds for any real number x. Half joking, all in earnest, one could say that previous identity is a nice gift from the NS principle.

On the other hand, it is important to note that, by adding the following two chained CH-type inequalities

$$P(A_K = +1, B_K = +1) - P(A_0 = +1, B_0 = +1)$$

$$- \sum_{k=1}^{K}\left[P(A_k = +1, B_{k-1} = -1) + P(A_{k-1} = -1, B_k = +1)\right] \overset{\mathrm{LR}}{\leq} 0, \quad (23a)$$

and

$$P(A_K = -1, B_K = -1) - P(A_0 = -1, B_0 = -1)$$

$$- \sum_{k=1}^{K}\left[P(A_k = -1, B_{k-1} = +1) + P(A_{k-1} = +1, B_k = -1)\right] \overset{\mathrm{LR}}{\leq} 0, \quad (23b)$$

we obtain the inequality

$$P^+(A_K, B_K) - P^+(A_0, B_0) - \sum_{k=1}^{K}\left[P^-(A_k, B_{k-1}) + P^-(A_{k-1}, B_k)\right] \overset{\mathrm{LR}}{\leq} 0,$$

which, in turn, can be readily converted into the chained CHSH-type inequality $\mathrm{CHSH}_K \overset{\mathrm{LR}}{\le} 2K$, and vice versa. Notice that the amount of violation of *both* CH inequalities (23a) and (23b) when the Hardy conditions (1)-(4) are fulfilled is $P_K \le 0$. (For the inequality (23b), this follows at once from relation (12).)

Consider now the sum of probabilities on the lhs of inequality (23a)

$$\mathrm{CH}_K \equiv P_K - P(A_0 = +1, B_0 = +1)$$
$$- \sum_{k=1}^{K} \big[P(A_k = +1, B_{k-1} = -1) + P(A_{k-1} = -1, B_k = +1) \big],$$

where we assume that the various probabilities $P(A_k = i, B_{k'} = j)$ satisfy the NS conditions (10) and (11) (apart from the usual non-negativity and normalization constraints), but are otherwise arbitrary. Then, by applying a procedure similar to that used in Sec. 2 to prove relation (8), it can be shown that

$$\mathrm{CHSH}_K = 2K + 4\mathrm{CH}_K, \quad K = 1, 2, 3, \dots. \tag{24}$$

From the Tsirelson bound (6), we therefore deduce that

$$\mathrm{CH}_K \overset{\mathrm{QM}}{\le} L_K, \tag{25}$$

where L_K is the upper limit in Eq. (9). In particular, for $K = 1$, we retrieve the well-known quantum mechanical bound $\mathrm{CH}_1 \overset{\mathrm{QM}}{\le} \frac{1}{2}(\sqrt{2} - 1)$ [24]. Evidently, when the Hardy conditions (2)-(4) are satisfied, the relations (24) and (25) reduce to relations (8) and (9), respectively.

Lastly, it is to be mentioned that Ahanj *et al.* [25] (see also Ref. [13]) derived an upper bound on the Hardy fraction, for $K = 1$, by applying a sufficient condition for violating the principle of information causality [26] (IC).[4] Under this condition, they found an upper bound given by $P_1 \le \frac{1}{2}(\sqrt{2} - 1)$. Note that this bound is the same as the resulting upper limit in Eq. (9) for $K = 1$. This coincidence, however, is not accidental. Indeed, the link between the two approaches becomes clear upon considering the following two facts: (i) All NS correlations which violate the Tsirelson bound $2\sqrt{2}$ also violate IC [26, 27]; (ii) By relation (8), we have that $\mathrm{CHSH}_1 > 2\sqrt{2}$ whenever $P_1 > \frac{1}{2}(\sqrt{2} - 1)$ [13]. We thus conclude that the set of NS correlations fulfilling all the Hardy conditions (1)-(4) with $\frac{1}{2}(\sqrt{2} - 1) < P_1 \le 0.5$, violate IC.

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[4] The IC principle states that communication of m classical bits causes information gain of at most m bits. The NS principle is just IC for $m = 0$.

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